# Math 105 - Assignment 2: Solutions

1. a) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

$$f_x = \frac{2x(x^2+y^2) - (x^2-y^2)(2x)}{(x^2+y^2)^2}$$ , quotient rule

$$= \frac{2x[(x^2+y^2) - (x^2-y^2)]}{(x^2+y^2)^2}$$

$$= \frac{4xy^2}{(x^2+y^2)^2}$$

$$f_y = \frac{-2y(x^2+y^2) - (x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2y[-(x^2+y^2) - (x^2-y^2)]}{(x^2+y^2)^2}$$

$$= \frac{-2yx^2}{(x^2+y^2)^2}$$

b) $g(x,y) = (\cos x)^{\sin y}$

$g_x = \sin y (\cos x)^{\sin y - 1} \cdot \frac{\partial}{\partial x}(\cos x)$ , by chain rule & $\frac{d}{dx} x^a = a x^{a-1}$

$= -\sin y (\cos x)^{\sin y - 1} \sin x$

$g_y = (\cos x)^{\sin y} \log(\cos x) \frac{\partial}{\partial y}(\sin y)$ , by chain rule and $\frac{d}{dx} a^x = a^x \log a$

$= (\cos x)^{\sin y} \log(\cos x) \cos y$

2. I first claim that

$$f_{yxx} \overset{(1)}{=} f_{xyx} \overset{(2)}{=} f_{xxy}$$

To do this we need to invoke Clairaut's theorem:

We are given that $f_x$ and $f_y$ are continuous and the second partials are defined. So by Clairaut's theorem:

$$f_{xy} = f_{yx}$$

Taking partials with respect to $x$ of both sides, we get

$$f_{xyx} = f_{yxx}$$

Thus proving (1).

To prove (2), we define

$$g(x,y) = f_x(x,y).$$

We have

$$g_x = \frac{\partial}{\partial x} f_x \qquad , \qquad g_y = \frac{\partial}{\partial y} f_x$$
$$= f_{xx} \qquad\qquad\qquad = f_{xy}$$

and

$$g_{xy} = \frac{\partial}{\partial y} g_x \qquad\qquad g_{yx} = \frac{\partial}{\partial x} g_y$$
$$= \frac{\partial}{\partial y} f_{xx} \qquad\qquad = \frac{\partial}{\partial x} f_{xy}$$
$$= f_{xxy} \qquad\qquad = f_{xyx}$$

Since we are given that $f_{xx}$, $f_{xy}$ are continuous, we have $g_x$, $g_y$ are continuous. Also since $f_{xxy}$, $f_{xyx}$ are defined we have $g_{xy}$, $g_{yx}$ are defined, so $g$ satisfies the conditions of Clairaut's theorem and

$$g_{xy} = g_{yx}$$

$$\Rightarrow f_{xxy} = f_{xyx}$$

Thus proving 2.

So

$$\left[f_{xyx}(-4,3)\right]^2 + 5 f_{yxx}(-4,3) = 6$$

$$\Rightarrow \left[f_{xxy}(-4,3)\right]^2 + 5 f_{xxy}(-4,3) = 6$$

$$\Rightarrow \left[f_{xxy}(-4,3)\right]^2 + 5 f_{xxy}(-4,3) - 6 = 0$$

$$\Rightarrow \left(f_{xxy}(-4,3) - 1\right)\left(f_{xxy}(-4,3) + 6\right) = 0$$

$$\Rightarrow f_{xxy}(-4,3) = 1 \text{ or } f_{xxy}(-4,3) = -6.$$

3. $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x + e^{\sqrt{\log(\pi)}}$

We must solve for the critical points thus $\nabla f = 0$. ie

$$\begin{cases} f_x = 6x^2 + 6y^2 - 150 = 0 \\ f_y = 12xy - 9y^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 + y^2 = 25 & (1) \\ 3y(4x - 3y) = 0 & (2) \end{cases}$$

(2) tells us either $y = 0$ or $4x - 3y = 0$

Case ① $y = 0$

(1) tell us $x^2 = 25$ or $x = \pm 5$

So we have the points $(5,0), (-5,0)$.

Case ② $4x - 3y = 0$. In this case we have

$$x = \frac{3y}{4}$$

(1) tells us

$$\left(\frac{3y}{4}\right)^2 + y^2 = 25$$

$$\Rightarrow y^2\left(\frac{9}{16} + 1\right) = 25$$

$$\Rightarrow y^2 \frac{25}{16} = 25$$

$$\Rightarrow y^2 = 16$$

$$\Rightarrow y = \pm 4.$$

If $y = 4$, then $x = \frac{3(4)}{4} = 3$

$y = -4$, then $x = \frac{3(-4)}{4} = -3$

Thus there are 4 critical points

$$(5,0),\ (-5,0),\ (3,4),\ (-3,-4).$$

To determine whether or not they are max, min or crit points, we compute the Hessian / discrimanant.

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(6x^2 + 6y^2 - 150) = 12x$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(6x^2 + 6y^2 - 150) = 12y$$

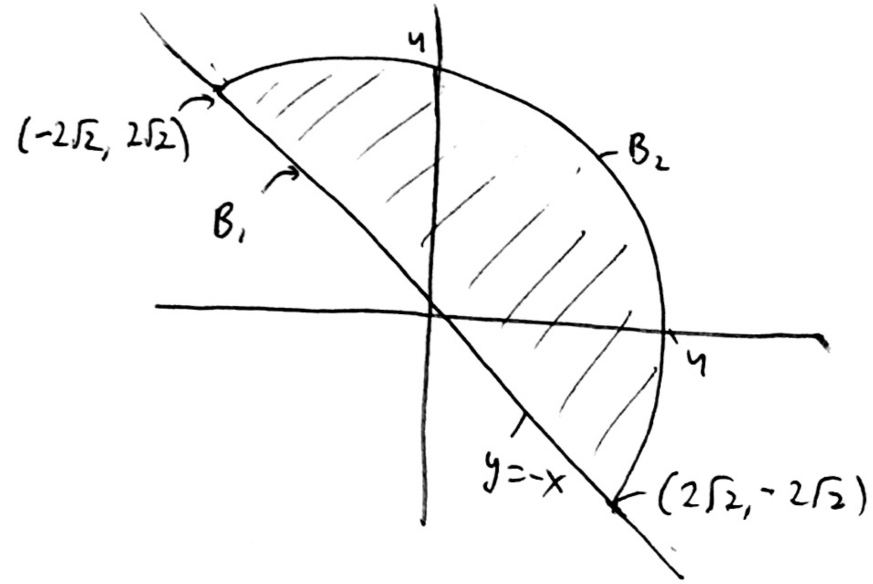
$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}(12xy - 9y^2) = 12x - 18y$$

$$f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}(12xy - 9y^2) = 12y$$

So
$$\begin{aligned} D &= f_{xx} f_{yy} - f_{xy} f_{yx} \\ &= (12x)(12x - 18y) - (12y)^2 \\ &= 144x^2 - 216xy - 144y^2 \end{aligned}$$

| $(a,b)$ | $D(a,b)$ | $f_{xx}(a,b)$ | Classification |
|---|---|---|---|
| $(5,0)$ | $3600 > 0$ | $60 > 0$ | local min |
| $(-5,0)$ | $3600 > 0$ | $-60 < 0$ | local max |
| $(3,4)$ | $-3600 < 0$ | — | saddle |
| $(-3,-4)$ | $-3600 < 0$ | — | saddle |

4. Let us first draw the region of interest.




It is the way it is because $x^2+y^2 \leq 16$ is the circle of radius 4 including the interior, $y \geq -x$ implies we are only interested in the region above the line $y=-x$.

To find the intersection points we need to know when the line $y=-x$ intersects the circle of radius 4.

$$x^2+(-x)^2=16$$

$$2x^2=16$$

$$\Rightarrow \quad x = \pm 2\sqrt{2}$$

So it intersects at the points $(2\sqrt{2}, -2\sqrt{2})$, $(-2\sqrt{2}, 2\sqrt{2})$

Let's call $B_1$ the straight part of the region of interest and lets call $B_2$ the semi-circular part of the region.

To optimize $f(x,y) = x^2 + 2y^2 - 4x$ on the semi-circle we need to

① Find the critical points inside the disc

② Find points of interest on $B_1$

③ Find points of interest on $B_2$.

① We find critical points:

$$f_x = 2x - 4 = 0 \quad \Rightarrow \quad x = 2$$

$$f_y = 4y \quad = 0 \quad \Rightarrow \quad y = 0$$

Since $(2,0)$ is inside the disc and $0 > -2$, we have $(0,2)$ is a critical point inside our region.

② Since on $B_1$ we have $y = -x$, and $x$ ranges from $-2\sqrt{2}$ to $2\sqrt{2}$,

we define
$$\begin{aligned} g(x) &= f(x,-x) \\ &= x^2 + 2(-x)^2 - 4x \\ &= 3x^2 - 4x \end{aligned}$$

$$g'(x) = 6x - 4 = 0$$

$$\Rightarrow x = \frac{2}{3}$$

So points of interes are $\left(\frac{2}{3}, \frac{-2}{3}\right)$ and the end points $(-2\sqrt{2}, 2\sqrt{2}), (2\sqrt{2}, -2\sqrt{2})$

③ On $B_3$ we use Lagrange multipliers.

Objective function: $f(x,y) = x^2 + 2y^2 - 4x$

Constraint: $g(x,y) = x^2 + y^2 - 16 = 0$, $-2\sqrt{2} \le x \le 2\sqrt{2}$.

So we have $\nabla f = \lambda \nabla g$, we gives us the following equations.

$$\begin{cases} 2x - 4 = 2\lambda x & (1) \\ 4y = 2\lambda y & (2) \\ x^2 + y^2 = 16 & (3) \end{cases}$$

(2) tell us $2y(2-\lambda) = 0$, so either $y=0$ or $2-\lambda=0$

Case ① $y=0$

(3) tells us $x^2 = 16$, so $x = \pm 4$

But since $-2\sqrt{2} \le x \le 2\sqrt{2}$, we have this case is not possible.

Case ② $2-\lambda = 0$ ie $\lambda = 2$.

(1) tells us. $2x - 4 = 2(2)x$

$\Rightarrow \quad 2x = -4$

$\Rightarrow \quad x = -2$.

$x = -2$ & (3) $\Rightarrow (-2)^2 + y^2 = 16$

$\Rightarrow \quad y^2 = 12$

$\Rightarrow \quad y = 2\sqrt{3}$

So we have to check the point $(-2, 2\sqrt{3})$ and the endpoints.

| $(x,y)$ | $f(x,y) = x^2+2y^2-4x$ |
|---|---|
| $(0,2)$ | $8$ |
| $(\frac{2}{3}, \frac{-2}{3})$ | $-\frac{4}{3}$ |
| $(-2\sqrt{2}, 2\sqrt{2})$ | $24+8\sqrt{2} \approx 35.3137$ |
| $(2\sqrt{2}, -2\sqrt{2})$ | $24-8\sqrt{2} \approx 12.6863$ |
| $(-2, 2\sqrt{3})$ | $36$ |

So the absolute maximum is $36$ at $(-2, 2\sqrt{3})$

absolute minimum is $-\frac{4}{3}$ at $(\frac{2}{3}, \frac{-2}{3})$

5. Given a point $(x,y)$ the distance from $(0,0)$ to $(x,y)$ is

$$\sqrt{x^2+y^2}$$

To minimize the distance between the curve $y^2x=16$ we use the method of Lagrange multipliers. Since square roots are hard to work with we will minimize the distance squared, since distance is alway positive and the square function is increasing, this is an equivalent problem.

Objective function: $f(x,y) = x^2+y^2$

Constraint : $g(x,y) = xy^2-16 = 0$

By Lagrange multipliers, $\nabla f = \lambda \nabla g$, ie

$$\begin{cases} 2x = \lambda y^2 & (1) \\ 2y = 2\lambda xy & (2) \\ xy^2 = 16 & (3) \end{cases}$$

Note that if either $x, y = 0$, then (3) cannot hold, so in particular we can divide by $x$ and $y$.

$$(1) \Rightarrow \quad \lambda = \frac{2x}{y^2}$$

$$(2) \Rightarrow \quad \lambda = \frac{2y}{2xy} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{2x}{y^2}, \text{ ie } \quad y^2 = 2x^2$$

Substituting $y^2 = 2x^2$ into (3) we get

$$x(2x^2) = 16$$

$$\Rightarrow \quad x^3 = 8$$

$$\Rightarrow \quad x = 2$$

$x = 2$ tells us $y^2 = 2(2)^2$
$= 8$

So $y = \pm 2\sqrt{2}$

The points of interest are $(2, 2\sqrt{2}), (2, -2\sqrt{2})$.

Since $f(2, \pm 2\sqrt{2}) = 12$, we have the minimum distance squared is 12. Thus the minimum distance is $\sqrt{12}$, ie $2\sqrt{3}$.